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Uniform Convergence of Fourier–Jacobi Series

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Necessary and sufficient conditions which imply the uniform convergence of the Fourier–Jacobi series of a continuous function are obtained under an assumption that the Fourier–Jacobi series is convergent at the end points of the segment of orthogonality $[-1, 1]$. The conditions are in terms of the modulus of continuity, Λ-variation, and the modulus of variation of a function. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

1. Throughout this paper we use the following general notations: N is the set of positive integers. By c we denote positive constants, possibly depending on some fixed parameters, and, in general, distinct in different formulas. Sometimes the important arguments of c will be written explicitly in the expressions for it. For quantities A_n and B_n, possibly depending on some other variables as well, we write $A_n = o(B_n)$, $A_n = O(B_n)$, or $A_n \asymp B_n$ as $n \to \infty$, if $\lim_{n\to\infty} A_n/B_n = 0$, $A_n \leqslant cB_n$, or $c_1 B_n \leqslant A_n \leqslant c_2 B_n$, $n \in N$, respectively.

$C[a, b]$ is the space of continuous functions on $[a, b]$ with uniform norm $\|\cdot\|_{[a,b]}$. $\omega(f, \delta, [a, b]) = \max\{|f(x) - f(t)| : x, t \in [a, b] \text{ and } |x - t| \leqslant \delta\}$ is the modulus of continuity of $f \in C[a, b]$ on $[a, b]$. $\omega(\delta)$ is a given modulus of continuity, i.e., a continuous non-decreasing semiadditive function on $[0, \infty)$, $\omega(0) = 0$. $H^\omega = \{f : \omega(f, \delta, [a, b]) = O(\omega(\delta)) \text{ for } \delta \geqslant 0\}$.

If a function g is integrable on $[-\pi, \pi]$, then g has a Fourier series with respect to the trigonometric system $(1, \cos n\theta, \sin n\theta)_{n=1}^{\infty}$, and we denote the nth partial sum of the Fourier series of g by $S_n(g, \cdot)$, i.e.,

$$S_n(g, \theta) = \frac{a_0(g)}{2} + \sum_{k=1}^{n} (a_k(g)\cos k\theta + b_k(g)\sin k\theta),$$

where

$$a_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\cos k\tau\, d\tau \qquad \text{and} \qquad b_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\sin k\tau\, d\tau$$

are the kth Fourier coefficients of the function g.

The function $\rho^{(\alpha,\beta)}$ is called a Jacobi weight if $\rho^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, where $\alpha > -1$ and $\beta > -1$. If $\rho^{(\alpha,\beta)}$ is a Jacobi weight, then by $\sigma(\rho^{(\alpha,\beta)}) = (P_n^{(\alpha,\beta)}(x))_{n=0}^{\infty}$ we denote the corresponding system of orthonormal polynomials $P_n^{(\alpha,\beta)}(x) = \gamma_n(\alpha,\beta)x^n +$ lower degree terms, $\gamma_n(\alpha,\beta) > 0$, i.e.,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)\rho^{(\alpha,\beta)}(t)\, dt = \delta_{nm}.$$

The system $\sigma(\rho^{(\alpha,\beta)})$ is defined uniquely and called the Jacobi system of orthonormal polynomials.

If $f\rho^{(\alpha,\beta)}$ is an integrable function on $[-1,1]$, then f has a Fourier series with respect to the system $\sigma(\rho^{(\alpha,\beta)})$, and by $S_n^{(\alpha,\beta)}(f,x)$ we denote the nth partial sum of the Fourier series of f with respect to the system $\sigma(\rho^{(\alpha,\beta)})$, i.e.,

$$S_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} a_k^{(\alpha,\beta)}(f)P_k^{(\alpha,\beta)}(x) = \int_{-1}^{1} f(t)K_n^{(\alpha,\beta)}(x,t)\rho^{(\alpha,\beta)}(t)\, dt, \tag{1}$$

where

$$a_k^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(t)P_k^{(\alpha,\beta)}(t)\rho^{(\alpha,\beta)}(t)\, dt$$

is the kth Fourier coefficient of the function f, and

$$K_n^{(\alpha,\beta)}(x,t) = \sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(t) \tag{2}$$

is the Dirichlet kernel of the system $\sigma(\rho^{(\alpha,\beta)})$.

By $U^{(\alpha,\beta)}$ we denote the class of functions, defined on the segment $[-1,1]$, for which the sequence of partial sums of its Fourier series with respect to the system $\sigma(\rho^{(\alpha,\beta)})$ is uniformly convergent on the whole segment of orthogonality $[-1,1]$, i.e., $\|S_n^{(\alpha,\beta)}(f,\cdot) - f\|_{[-1,1]} = o(1)$ as $n \to \infty$.

We say that a function f, defined on the segment $[-1,1]$, belongs to the class $EP^{(\alpha,\beta)}$, if the sequences $(S_n^{(\alpha,\beta)}(f,\pm 1))_{n=0}^{\infty}$ are convergent.

Definition 1. We say that a function f, defined on the segment $[a,b]$, belongs to $DL[a,b]$ class, if $\omega(f,1/n,[a,b])\ln n = o(1)$ as $n \to \infty$.

Definition 2 (Waterman [18]). Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$. A function f is said to have Λ-bounded variation on $[a,b]$, i.e., $f \in \Lambda BV[a,b]$, if

$$v_\Lambda(f,[a,b]) = \sup_{\Pi} \sum_{k=1}^{n} \frac{|f(x_{2k}) - f(x_{2k-1})|}{\lambda_k} < \infty,$$

where Π is an arbitrary system of disjoint intervals $(x_{2k-1}, x_{2k}) \subset [a,b]$, $k = 1, 2, \dots, n$.

If $\lambda_k = 1$, $k \in N$, then $\Lambda BV[a,b] = V[a,b]$, the Jordan class of functions of bounded variation. Following Waterman, we say that f is of *harmonic* bounded variation, i.e., $f \in HBV[a,b]$, if $\lambda_k = k$, $k \in N$.

Definition 3 (Waterman [19]). Let $\Lambda(\ell) = (\lambda_{k+\ell})_{k=1}^{\infty}$, $\ell \in N$, where the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$ satisfies the condition of Definition 2. A function $f \in \Lambda BV[a,b]$ is said to be continuous in Λ-variation, i.e., $f \in \Lambda_C BV[a,b]$, if $v_{\Lambda(\ell)}(f,[a,b]) = o(1)$ as $\ell \to \infty$.

Definition 4 (Čanturija [9]). Let f be a bounded function on $[a,b]$. The modulus of variation of f is called the function $v(f,n,[a,b])$ defined for $n = 0, 1, 2, \dots$ as follows: $v(f,0,[a,b]) = 0$, while for $n \geqslant 1$

$$v(f,n,[a,b]) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(x_{2k}) - f(x_{2k-1})|,$$

where Π_n is an arbitrary system of n disjoint subintervals (x_{2k-1}, x_{2k}), $k = 1, \dots, n$, of the segment $[a,b]$.

If $v(n)$, $n \in N$, is a non-decreasing convex function and $v(0) = 0$, then we call $v(n)$ the modulus of variation.

The class of functions which satisfy the relation $v(f,n,[a,b]) = O(v(n))$ as $n \to \infty$ will be denoted by $V[v][a,b]$.

In particular, $V[1][a,b] = V[a,b]$.

If there is no ambiguity, we will usually omit the dependence on the domain and simply refer to one of the introduced classes of functions or the quantities as C, $V_\Lambda, \dots$, or $v_\Lambda(f)$, $v(f,n)$, etc.

2. The well-known result [17, Theorem 9.1.2, p. 246] about equiconvergence indicates that uniform convergence conditions of Fourier–Jacobi series strictly inside of the orthogonality segment, i.e., on an arbitrary segment $[a,b] \subset (-1,1)$, should be similar to uniform convergence conditions of Fourier series with respect to the trigonometric system. For example, the condition $f \in DL$ guarantees the uniform convergence of Fourier–Jacobi series of the function f on $[a,b] \subset (-1,1)$ (cf. [16, Theorem 4.7, pp. 146, 300]). Let us recall that $f \in DL$ is also a sufficient condition for the uniform convergence of a 2π-periodic function's Fourier series with respect to the trigonometric system.

It is also known that $DL \subset U^{(\alpha,\beta)}$ when $-1<\alpha\leqslant -1/2$ and $-1<\beta\leqslant -1/2$ [4, Theorem 1, p. 947]. The same theorem implies that for $-1<\alpha< -1/2$ and $-1<\beta< -1/2$ only continuity of a function f guarantees that $f \in EP^{(\alpha,\beta)}$.

On the other hand, for the uniform convergence of Fourier–Jacobi series on the whole segment of orthogonality far stronger conditions must be imposed on a function (cf. [1, 4, 12]).

THEOREM (Agakhanov and Natanson [1]). *Let $\alpha > -1/2$ and $\beta > -1/2$. If the modulus of continuity of a function f satisfies the condition*

$$\lim_{n\to\infty} \omega\left(f, \frac{1}{n}\right) n^{\max(\alpha,\beta)+1/2} = 0, \tag{3}$$

then $f \in U^{(\alpha,\beta)}$.

Let us mention, that condition (3) is necessary for the convergence of the Fourier–Jacobi series at the end points of the segment $[-1, 1]$ as well.

Summarizing all the above the following hypothesis arises: Let the Fourier–Jacobi series of a continuous function f be convergent at the end points of the segment $[-1, 1]$. In addition, let the function satisfy a condition implying the uniform convergence of its Fourier series with respect to the trigonometric system. (It is a far less restrictive condition than a condition guaranteeing the uniform convergence of its Fourier–Jacobi series on the whole segment $[-1, 1]$.) Do these conditions guarantee the uniform convergence of the Fourier–Jacobi series of the function f on the whole segment $[-1, 1]$?

The first paper dealing with this problem is due to Zorshchikov.

THEOREM (Zorshchikov [22]). *Let a function f be representable in the form $f(x) = (1 - x^2)h(x)$ and $h \in DL$. If $f \in EP^{(\alpha,\beta)}$ for some $-1/2\leqslant\alpha\leqslant 1/2$ and $-1/2\leqslant\beta\leqslant 1/2$, then $f \in U^{(\alpha,\beta)}$.*

Belen'kii has completely solved the problem in terms of the modulus of continuity.

THEOREM (Belen'kii [8, Theorem, p. 901]). *Let $\alpha > -1$ and $\beta > -1$. Then the inclusion*

$$DL \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$$

is valid.

In the present paper, we study those conditions on the variation of a continuous function which guarantee the uniform convergence of its Fourier–Jacobi series under an assumption that the series is convergent at the end points of the segment of orthogonality $[-1, 1]$.

2. MAIN RESULTS

In what follows, we always assume that indices $\alpha > -1$ and $\beta > -1$ of a weight function $\rho^{(\alpha,\beta)}$ are arbitrary but fixed. In addition, sometimes we abbreviate notations for the intersection of two sets. For example, we write $CV[\upsilon]$ instead of $C \cap V[\upsilon]$.

THEOREM 1. *Let H^ω and $V[\upsilon]$ be classes of functions defined by a modulus of continuity $\omega(\delta)$ and a modulus of variation $\upsilon(n)$, respectively. Then the inclusion*

$$H^\omega V[\upsilon] \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$$

is valid if and only if

$$\lim_{n\to\infty} \min_{1\leqslant \ell \leqslant n} \left\{ \omega\left(\frac{1}{n}\right) \sum_{k=1}^{\ell} \frac{1}{k} + \sum_{k=\ell+1}^{n-1} \frac{\upsilon(k)}{k^2} \right\} = 0. \tag{4}$$

COROLLARY 1. *Let*

$$\omega(\delta) = \frac{(\ln\ln 1/\delta)^\gamma}{\ln 1/\delta}, \quad \gamma \geqslant 0 \qquad \text{and} \qquad \upsilon(n) = \frac{n}{\ln n \ln\ln n}. \tag{5}$$

Then $H^\omega V[\upsilon] \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$.

COROLLARY 2. *Let $V[\upsilon]$ be a class of functions defined by a given modulus of variation $\upsilon(n)$. Then the inclusion*

$$CV[\upsilon] \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$$

is valid if and only if

$$\sum_{k=1}^{\infty} \frac{\upsilon(k)}{k^2} < \infty. \tag{6}$$

THEOREM 2. *Let ΛBV be a class of functions defined by a given sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then the inclusion*

$$C\Lambda BV \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$$

is valid if and only if

$$\Lambda BV \subset HBV. \tag{7}$$

In particular, $CV \cap EP^{(\alpha,\beta)} \subset U^{(\alpha,\beta)}$.

Let us clarify the significance of condition (4) and outline the central idea of the proofs.

Condition (4) combines conditions imposed on the modulus of continuity and the variation of a function. As a result (see Corollary 1) it implies convergence conditions which are less restrictive than conditions imposed only on the modulus of continuity or the variation of a function, thus the convergence is obtained for wider classes of functions.

The techniques used to obtain a Fourier series convergence condition in terms of the modulus of continuity are typically based on the following inequality (or some variation of it) [16, p. 35]:

$$|f(x) - S_n^{(\alpha,\beta)}(f,x)| \leqslant (1 + L_n^{(\alpha,\beta)}(x))E_n(f) \tag{8}$$

for $x \in [-1,1]$, where $E_n(f) = \inf_{P_n} \|f - P_n\|$ is the best polynomial approximation for f of degree less than n in the uniform metric and $L_n^{(\alpha,\beta)}(x) = \int_{-1}^{1} |K^{(\alpha,\beta)}(x,t)| \rho^{(\alpha,\beta)}(t)\,dt$ is the nth Lebesgue constant of the Fourier–Jacobi series.

By Jackson's inequality, $E_n(f) \leqslant 11\omega(f, 1/n)$ [16, p. 391]. Thus, by virtue of condition (8), $\lim_{n\to\infty} L_n^{(\alpha,\beta)}(x)\omega(f,1/n) = 0$ guarantees the convergence of Fourier–Jacobi series at the given point x. Hence, given an accurate estimate for the nth Lebesgue constant (cf. [4]), convergence conditions of Fourier–Jacobi series in term of the modulus of continuity will instantly follow.

However, in order to obtain a convergence condition in terms of the variation of a function, much more delicate estimate of the Lebesgue constant is needed. Namely, an estimate for $\int_{x_k}^{x_{k+1}} |K^{(\alpha,\beta)}(x,t)| \rho^{(\alpha,\beta)}(t)\,dt$, $| \int_{-1}^{x_k} K^{(\alpha,\beta)}(x,t) \rho^{(\alpha,\beta)}(t)\,dt|$, and $| \int_{x_k}^{1} K^{(\alpha,\beta)}(x,t) \rho^{(\alpha,\beta)}(t)\,dt|$, where $x_k \asymp \cos(k/n)$, $k = 1, 2, \ldots, n-1$.

This is an outline of the proof: In Lemma 1 we obtain estimates for the Lebesgue constant over a subinterval with the desired length. Next, in Lemma 2 we estimate the tail of the Fourier–Jacobi series of a given function it terms of the functions oscillation over the system of non-overlapping intervals. Then the actual proofs of the theorems follow the well-known schemes.

Remark. Let us also mention that the conditions imposed on a function in Theorems 1 and 2 and Corollary 2 are necessary and sufficient for the uniform convergence of its Fourier series with respect to the trigonometric system. Regarding conditions (4) and (7) see [10, Theorem 1, p. 476; 18, Theorem 2, p. 112]. Theorems 1 and 2 as a corollary imply Theorem B as well as necessary and sufficient conditions in terms of Φ-variation [21] and Banach indicatrix [7] (see [10, Corollaries 2, 3, p. 478; 11, 13, Theorem 1, p. 620; 15]).

3. PRELIMINARIES

In what follows, we always assume that the integers r and m are determined by the following conditions: $\alpha \in (r - 3/2, r - 1/2]$ and $\beta \in (m - 3/2, m - 1/2]$. $n \in N$ assumed to be sufficiently large.

In addition, $g(\tau) \equiv f(\cos \tau)$ for $\tau \in [0, \pi]$, where g is a 2π-periodic even function. $\theta = \arccos x$ and $\tau = \arccos t$ for $x, t \in [-1, 1]$.

Let us mention that the function g belongs to the same class of a generalized variation whatever class the function f belongs to and $\omega(\delta, g) \leqslant \omega(\delta, f)$ for $\delta \geqslant 0$.

We will use the well-known estimates:

$$\sum_{k=1}^{n} k^{\gamma} = O(n^{\gamma+1}) \quad \text{for } \gamma > -1 \tag{9}$$

and

$$\sum_{k=n}^{\infty} \frac{1}{k^{\gamma}} = O\left(\frac{1}{n^{\gamma-1}}\right) \quad \text{for } \gamma > 1. \tag{10}$$

The following formulas and lemmas are necessary in what follows:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{11}$$

$$|P_{n-1}^{(\alpha,\beta)}(x)| < c(\alpha, \beta)\left((1-x)^{1/2} + \frac{1}{n}\right)^{-\alpha-1/2}\left((1+x)^{1/2} + \frac{1}{n}\right)^{-\beta-1/2} \tag{12}$$

holds for $x \in [-1, 1]$ and $n \in N$.

$$P_n^{(\alpha,\beta)}(\cos \tau) = \kappa(\alpha, \beta, \tau)[\cos(\tilde{n}\tau + \tilde{\gamma}) + O(1)(n \sin \tau)^{-1}], \tag{13}$$

where $\kappa(\alpha, \beta, \tau) = 2^{-(\alpha+\beta)/2}\pi^{-1/2}\sin^{-\alpha-1/2}(\tau/2)\cos^{-\beta-1/2}(\tau/2)$, $\tilde{n} = n + (\alpha + \beta + 1)/2$, $\tilde{\gamma} = -(2\alpha + 1)\pi/4$, and $c/n \leqslant \tau \leqslant \pi - c/n$:

$$(x - t)K_n^{(\alpha,\beta)}(x, t) = v_n^{(\alpha,\beta)}(P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(t) - P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(t)) \tag{14}$$

and

$$(x - t)K_n^{(\alpha,\beta)}(x, t) = \mu_n^{(\alpha,\beta)}((1-t)P_n^{(\alpha+1,\beta)}(t)P_n^{(\alpha,\beta)}(x) - (1-x)P_n^{(\alpha+1,\beta)}(x)P_n^{(\alpha,\beta)}(t)), \tag{15}$$

where $0 < \lim_{n\to\infty} v_n^{(\alpha,\beta)} < \infty$ and $0 < \lim_{n\to\infty} \mu_n^{(\alpha,\beta)} < \infty$.

Regarding (11)– (15) see [17, formula (4.1.3), p. 59], [17, Theorem 7.32.2, p. 169], [17, Theorem 8.21.13, p. 197], [17, formula (4.5.2), p. 71], and [8, p. 903], respectively.

In addition, let us introduce the following notations:

$$K_1 \equiv K_1(\alpha,\beta,n,\theta,\tau)$$
$$= 2^{\alpha+\beta+r+m+2}\mu_{n-r-m}^{(\alpha+r,\beta+m)} \sin^{2r}(\theta/2)\cos^{2m}(\theta/2)P_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta)$$
$$\frac{P_{n-r-m}^{(\alpha+r+1,\beta+m)}(\cos\tau)}{\cos\theta-\cos\tau}\sin^{2\alpha+3}(\tau/2)\cos^{2\beta+1}(\tau/2) \tag{16}$$

and

$$K_2 \equiv K_2(\alpha,\beta,n,\theta,\tau)$$
$$= 2^{\alpha+\beta+r+m+2}\mu_{n-r-m}^{(\alpha+r,\beta+m)} \sin^{2r+2}(\theta/2)\cos^{2m}(\theta/2)P_{n-r-m}^{(\alpha+r+1,\beta+m)}(\cos\theta)$$
$$\frac{P_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\tau)}{\cos\theta-\cos\tau}\sin^{2\alpha+1}(\tau/2)\cos^{2\beta+1}(\tau/2). \tag{17}$$

It is trivial to check that (see (15))

$$(1-\cos\theta)^r(1+\cos\theta)^m K_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta,\cos\tau)\rho^{(\alpha,\beta)}(\cos\tau)\sin\tau = K_1 - K_2. \tag{18}$$

Furthermore, let

$$t_n^i(k) \equiv \frac{\pi k-\gamma_i}{n_i}, \tag{19}$$

where $n_1 = n-r-m+(\alpha+r+\beta+m+2)/2$, $\gamma_1 = -(2\alpha+2r+3)\pi/4$, $n_2 = n-r-m+(\alpha+r+\beta+m+1)/2$, and $\gamma_2 = -(2\alpha+2r+1)\pi/4$, respectively. In addition, $t_n^i(0)\equiv 0$ and $t_n^i(\bar n)\equiv \pi/2$, where $\bar n = \min(\bar n_1,\bar n_2)$ and $\bar n_i$ is the largest integer satisfying the condition $t_n^i(\bar n_i)\leqslant \pi/2$, $i=1,2$.

It is obvious that

$$t_n^i(k) \asymp \frac{k}{n} \tag{20}$$

for $i=1,2$.

Let us set

$$d_k^i \equiv d_{k,n}^i(\theta) \equiv \int_{t_n^i(k-1)}^{t_n^i(k)} |K_i|\,d\tau, \tag{21}$$

$$\triangle_k^i \equiv \Delta_{k,n}^i(\theta) \equiv \left|\int_{t_n^i(k)}^{\pi/2} K_i\,d\tau\right|, \tag{22}$$

and

$$\nabla_k^i \equiv \nabla_{k,n}^i(\theta) \equiv \left| \int_{t_n^i([p/2])}^{t_n^i(k)} K_i \, d\tau \right| \tag{23}$$

for $i = 1, 2$, $n \in N$, $k = 1, 2, \ldots, n$, where $p \in N$ is defined by condition (24). (Here and elsewhere $[a]$ means the integer part of a number a.)

LEMMA 1. *Let*

$$\theta \in [0, \pi/3] \cap [t_n^1(p-1), t_n^1(p)] \tag{24}$$

for some $p \in N$. *Then the estimates*

$$d_k^i = \frac{O(1)}{|p-k|}, \quad k = [p/2], [p/2]+1, \ldots, \bar{n}, \quad k \neq p-1, p, p+1,$$

$$\triangle_k^i = \frac{O(1)}{k-p}, \quad k = p+2, \ldots, \bar{n},$$

$$\nabla_k^i = \frac{O(1)}{p-k}, \quad k = [p/2], [p/2]+1, \ldots, p-2,$$

hold for $i = 1, 2$.

Proof. Let us mention that by (20) and (24), for $k = [p/2], [p/2] + 1, \ldots, \bar{n}$, we have

$$\int_{t_n^i(k-1)}^{t_n^i(k)} \frac{\sin^\alpha(\tau/2)\cos^\beta(\tau/2)}{|\cos\theta - \cos\tau|} \, d\tau = O(1) \frac{1}{n} \left(\frac{k}{n}\right)^\alpha \frac{n^2}{|p^2 - k^2|}, \qquad k \neq p-1, p, p+1, \tag{25}$$

$$\sin^\alpha(\theta/2) = O(1) \left(\frac{p}{n}\right)^\alpha \qquad \text{and} \qquad \cos^\beta(\theta/2) = O(1). \tag{26}$$

Next, let $k \neq p-1, p, p+1$ and $k \leqslant \bar{n}$. Then by virtue of (12), (16), (21), (25), and (26) we obtain

$$\begin{aligned} d_k^1 &= \int_{t_n^1(k-1)}^{t_n^1(k)} |K_1| \, d\tau \\ &= O(1) \sin^{2r}(\theta/2)\cos^{2m}(\theta/2)\sin^{-\alpha-r-1/2}(\theta/2)\cos^{-\beta-m-1/2}(\theta/2) \\ &\quad \int_{t_n^1(k-1)}^{t_n^1(k)} \frac{\sin^{-\alpha-r-3/2}(\tau/2)\cos^{-\beta-m-1/2}(\tau/2)}{|\cos\theta - \cos\tau|} \sin^{2\alpha+3}(\tau/2)\cos^{2\beta+1}(\tau/2) \, d\tau \end{aligned} \tag{27}$$

$$= O(1)\left(\frac{p}{n}\right)^{r-\alpha-1/2}\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha-r+3/2}\frac{n^2}{|p^2-k^2|}$$

$$= O(1)\frac{k^{\alpha-r+3/2}}{p^{\alpha-r+1/2}|p^2-k^2|} = O(1)\frac{k^{\alpha-r+3/2}}{p^{\alpha-r+1/2}(p+k)|p-k|} = \frac{O(1)}{|p-k|}.$$

Indeed, $k^{\alpha-r+3/2}p^{-\alpha+r-1/2}/(p+k) \leqslant \min((k/p)^{\alpha-r+3/2}, (p/k)^{-\alpha+r-1/2})$. However, $\alpha - r + 3/2 > 0$ and $-\alpha + r - 1/2 \geqslant 0$. So the first or the second expression in "min" will be bounded by 1, depending whether $k < p$ or $p < k$, respectively.

We use asymptotic formula (13) in order to estimate $\triangle_k^1$. By virtue of (12), (16), and (22) for $k = p+2, \ldots, \bar{n}$ we have

$$\begin{aligned}
\triangle_k^1 &= \left| \int_{t_n^1(k)}^{\pi/2} K_1 \, d\tau \right| \\
&= O(1)\sin^{2r}(\theta/2)\cos^{2m}(\theta/2)\sin^{-\alpha-r-1/2}(\theta/2)\cos^{-\beta-m-1/2}(\theta/2) \\
&\quad \left\{ \left| \int_{t_n^1(k)}^{\pi/2} \cos(n_1\tau + \gamma_1)\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\theta - \cos\tau} d\tau \right| \right. \\
&\quad \left. + \frac{O(1)}{n}\int_{t_n^1(k)}^{\pi/2} \frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{\cos\theta - \cos\tau} d\tau \right\} \\
&\equiv J_1 + J_2. \qquad (28)
\end{aligned}$$

Next, let us mention that the function $y(\tau) = \cos(n_1\tau + \gamma_1)$ has opposite and constant sign on neighbor segments $[t_n^1(k-1), t_n^1(k)]$, $k \in N$ (see (19)), and the function $y(\tau) = \sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)/(\cos\theta - \cos\tau)$ is decreasing on the segment $[t_n^1(p+2), \pi/2]$. Thus,

$$\left| \int_{t_n^1(k)}^{\pi/2} \cos(n_1\tau + \gamma_1)\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\theta - \cos\tau} d\tau \right|$$

$$\leqslant \int_{t_n^1(k)}^{t_n^1(k+1)} \frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\theta - \cos\tau} d\tau.$$

Consequently, taking into account (25)–(28), we obtain

$$J_1 \leqslant O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2)$$

$$\int_{t_n^1(k)}^{t_n^1(k+1)} \frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\theta-\cos\tau}\,d\tau = \frac{O(1)}{k-p}. \tag{29}$$

As regards J_2, by virtue of (10), (25)–(27), and since $\alpha - r - 3/2 \leqslant -2$ we get

$$\begin{aligned}
J_2 &= O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2) \\
&\quad \frac{1}{n}\sum_{s=k}^{\bar{n}-1}\int_{t_n^1(s)}^{t_n^1(s+1)} \frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{\cos\theta-\cos\tau}\,d\tau \\
&= O(1)\left(\frac{p}{n}\right)^{r-\alpha-1/2}\frac{1}{n}\sum_{s=k}^{\bar{n}-1}\frac{1}{n}\left(\frac{s}{n}\right)^{\alpha-r+1/2}\frac{n^2}{s^2-p^2} \\
&= O(1)\frac{1}{p^{\alpha-r+1/2}}\sum_{s=k}^{\bar{n}-1}\frac{s^{\alpha-r+1/2}}{s^2-p^2} = O(1)\frac{1}{p^{\alpha-r+1/2}}\sum_{s=k}^{\bar{n}-1}\frac{s^2}{s^2-p^2}s^{\alpha-r-3/2} \\
&< O(1)\frac{1}{p^{\alpha-r+1/2}}\frac{k^2}{k^2-p^2}\sum_{s=k}^{\infty}s^{\alpha-r-3/2} = O(1)\frac{1}{p^{\alpha-r+1/2}}\frac{k^2}{k^2-p^2}k^{\alpha-r-1/2} \\
&= O(1)\frac{k^{\alpha-r+3/2}}{p^{\alpha-r+1/2}(k^2-p^2)} = \frac{O(1)}{k-p}.
\end{aligned} \tag{30}$$

Combination of (29) and (30) leads to the desired estimate.

Analogously, we estimate ∇_k^1. (See (13), (16), and (23).)

$$\begin{aligned}
\nabla_k^1 &= \left|\int_{t_n^1([p/2])}^{t_n^1(k)} K_1\,d\tau\right| \\
&= O(1)\sin^{2r}(\theta/2)\cos^{2m}(\theta/2)P_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta) \\
&\quad \left\{\left|\int_{t_n^1([p/2])}^{t_n^1(k)} \cos(n_1\tau+\gamma_1)\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\tau-\cos\theta}\,d\tau\right|\right.
\end{aligned}$$

$$+\frac{O(1)}{n}\int_{t_n^1([p/2])}^{t_n^1(k)}\frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{|\cos\tau-\cos\theta|}d\tau\Bigg\}$$

$$\equiv J^1+J^2. \tag{31}$$

Again, the function $y(\tau)=\cos(n_1\tau+\gamma_1)$ has opposite and constant sign on neighbor segments $[t_n^1(k-1),t_n^1(k)]$, $k\in N$, (see (19)) and the function $y(\tau)=\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)/(\cos\tau-\cos\theta)$ is increasing on the segment $[0,t_n^1(p-2)]$. Hence by virtue of (12) and (25)–(27), we have

$$J^1\leqslant O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2)\int_{t_n^1(k-1)}^{t_n^1(k)}\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\tau-\cos\theta}d\tau$$

$$=O(1)\left(\frac{p}{n}\right)^{r-\alpha-1/2}\frac{1}{n}\left(\frac{k}{n}\right)^{\alpha-r+3/2}\frac{n^2}{p^2-k^2}=O(1)\frac{k^{\alpha-r+3/2}}{p^{\alpha-k+1/2}(p^2-k^2)}=\frac{O(1)}{p-k}. \tag{32}$$

Similarly, due to (9), (12), (25)–(27), and since $\alpha-r+1/2>-1$

$$J^2=O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2)$$

$$\frac{1}{n}\sum_{s=[p/2]}^{k-1}\int_{t_n^1(s)}^{t_n^1(s+1)}\frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{\cos\tau-\cos\theta}d\tau$$

$$<O(1)\left(\frac{p}{n}\right)^{r-\alpha-1/2}\frac{1}{n}\sum_{s=1}^{k-1}\frac{1}{n}\left(\frac{s}{n}\right)^{\alpha-r+1/2}\frac{n^2}{p^2-s^2}$$

$$=O(1)\frac{1}{p^{\alpha-r+1/2}}\sum_{s=1}^{k-1}\frac{s^{\alpha-r+1/2}}{p^2-s^2}<O(1)\frac{1}{p^{\alpha-r+1/2}}\frac{1}{p^2-k^2}\sum_{s=1}^{k-1}s^{\alpha-r+1/2}$$

$$=O(1)\frac{k^{\alpha-r+3/2}}{p^{\alpha-r+1/2}(p^2-k^2)}=\frac{O(1)}{p-k}. \tag{33}$$

Finally, combination of (31)–(33) completes estimation for ∇_k^1. Estimates for d_k^2, $\triangle_k^2$, and ∇_k^2 are obtained analogously.

LEMMA 2. *Let the segments* $I_k^i\equiv I_{k,n}^i(f)\equiv[\tau_n^i(k),t_n^i(k)]$ ($f\in C$, $i=1,2$, $k=1,2,\ldots,n$, *and* $n\in N$) *be determined by the conditions*

$$|g(I_{k,n}^i)|\equiv|g(\tau_n^i(k))-g(t_n^i(k))|=\max_{\tau\in[t_n^i(k-1),t_n^i(k)]}|g(\tau)-g(t_n^i(k))|. \tag{34}$$

Then for every $f \in C$ *the estimate*

$$\left|(1-x)^r(1+x)^m \int_{-1}^{1} (f(x)-f(t))K_{n-r-m}^{(\alpha+r,\beta+m)}(x,t)\rho^{(\alpha,\beta)}(t)\,dt\right|$$

$$< c(\alpha,\beta)\left\{o(1)+\sum_{i=1}^{2}\sum_{k=1}^{n}\frac{|g(I^i_{n_k,n})|}{k}\right\} \tag{35}$$

holds uniformly with respect to $x \in [1/2, 1]$, *where* $|g(I^i_{n_k,n})| \geqslant |g(I^i_{n_{k+1},n})|$ *for* $k = 1, 2, \ldots, n-1$ *and* $i = 1, 2$.

Proof. Let $p \in N$ be determined by condition (24). Next,

$$\left|(1-x)^r(1+x)^m \int_{-1}^{1} (f(x)-f(t))K_{n-r-m}^{(\alpha+r,\beta+m)}(x,t)\rho^{(\alpha,\beta)}(t)\,dt\right|$$

$$= |(1-\cos\theta)^r(1+\cos\theta)^m$$

$$\left.\int_0^{\pi}(g(\tau)-g(\theta))K_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta,\cos\tau)\rho^{(\alpha,\beta)}(\cos\tau)\sin\tau\,d\tau\right|$$

$$\leqslant \left|\int_0^{t_n^1([p/2])}\right| + \left|\int_{t_n^1([p/2])}^{t_n^1(p-2)}\right| + \left|\int_{t_n^1(p-2)}^{t_n^1(p+1)}\right| + \left|\int_{t_n^1(p+1)}^{\pi/2}\right| + \left|\int_{\pi/2}^{\pi}\right|$$

$$\equiv J_1 + J_2 + J_3 + J_4 + J_5. \tag{36}$$

Obviously, the terms J_1 and J_2 will be absent if $p \leqslant 2$.

By virtue of the definition of modulus of continuity we obtain

$$J_3 \leqslant (1-\cos\theta)^r(1+\cos\theta)^m \int_{t_n^1(p-2)}^{t_n^1(p+1)} |g(\tau)-g(\theta)|$$

$$|K_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta,\cos\tau)|\rho^{(\alpha,\beta)}(\cos\tau)\sin\tau\,d\tau$$

$$\leqslant \omega(g, |t_n^1(p+1)-t_n^1(p-2)|)(1-\cos\theta)^r(1+\cos\theta)^m$$

$$\int_{t_n^1(p-2)}^{t_n^1(p+1)} |K_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta,\cos\tau)|\rho^{(\alpha,\beta)}(\cos\tau)\sin\tau\,d\tau = o(1). \tag{37}$$

Indeed, $\omega(g, |t_n^1(p+1)-t_n^1(p-2)|) = O(1)\omega(g, 1/n) = o(1)$. As regards the rest of the expression, it is bounded by a constant, uniformly with respect to $n \in N$ and $x \in [1/2, 1]$, due to formula (2) and estimates (9), (12), (25), and (26)

$$(1-\cos\theta)^r(1+\cos\theta)^m\int_{t_n^1(p-2)}^{t_n^1(p+1)}\left|\sum_{k=0}^{n-r-m}P_k^{(\alpha+r,\beta+m)}(\cos\theta)P_k^{(\alpha+r,\beta+m)}(\cos\tau)\right|$$

$$\rho^{(\alpha,\beta)}(\cos\tau)\sin\tau\,d\tau$$

$$=O(1)\sin^{2r}(\theta/2)\int_{t_n^1(p-2)}^{t_n^1(p+1)}\sum_{k=1}^{n-r-m}\left(\sin(\theta/2)+\frac{1}{k}\right)^{-\alpha-r-1/2}$$

$$\left(\sin(\tau/2)+\frac{1}{k}\right)^{-\alpha-r-1/2}\sin^{2\alpha+1}(\tau/2)\,d\tau$$

$$=O(1)\left(\frac{p}{n}\right)^{2\alpha+2r+1}\frac{1}{n}\left\{\sum_{k:\frac{1}{k}>\frac{p}{n}}\left(\frac{1}{k}\right)^{-2\alpha-2r-1}+\sum_{k:\frac{1}{k}\leqslant\frac{p}{n}}\left(\frac{p}{n}\right)^{-2\alpha-2r-1}\right\}$$

$$=O(1)\left(\frac{p}{n}\right)^{2\alpha+2r+1}\frac{1}{n}\left\{\left(\frac{n}{p}\right)^{2\alpha+2r+2}+n\left(\frac{p}{n}\right)^{-2\alpha-2r-1}\right\}=O(1).$$

In view of (12), (14), Lemma 5 [8, p. 904], and due to $|x-t|\geqslant 1/2$

$$J_5\leqslant O(1)\Bigg((1-x)^r(1+x)^m|P_{n-r-m}^{(\alpha+r,\beta+m)}(x)|\left|\int_{-1}^{0}\frac{f(t)-f(x)}{x-t}P_{n+1-r-m}^{(\alpha+r,\beta+m)}(t)\rho^{(\alpha,\beta)}(t)\,dt\right|$$
$$+(1-x)^r(1+x)^m|P_{n+1-r-m}^{(\alpha+r,\beta+m)}(x)|\left|\int_{-1}^{0}\frac{f(t)-f(x)}{x-t}P_{n-r-m}^{(\alpha+r,\beta+m)}(t)\rho^{(\alpha,\beta)}(t)\,dt\right|\Bigg)$$
$$=o(1) \tag{38}$$

uniformly with respect to $x\in[1/2,1]$.

By virtue of (16)–(18) we have

$$J_1\leqslant\left|\int_0^{t_n^1([p/2])}(g(\tau)-g(\theta))K_1\,d\tau\right|+\left|\int_0^{t_n^1([p/2])}(g(\tau)-g(\theta))K_2\,d\tau\right|\equiv J_{1,1}+J_{1,2}. \tag{39}$$

Furthermore, by (13) we get

$$J_{1,1} = \left| \int_0^{t_n^1([p/2])} (g(\tau) - g(\theta))K_1 \, d\tau \right|$$

$$= O(1)\sin^{2r}(\theta/2)\cos^{2m}(\theta/2)P_{n-r-m}^{(\alpha+r,\beta+m)}(\cos\theta)$$

$$\left\{ \left| \int_{t_n^1(1)}^{t_n^1([p/2])} (g(\tau) - g(\theta))\cos(n_1\tau + \gamma_1)\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\tau - \cos\theta} d\tau \right| \right.$$

$$\left. + \frac{O(1)}{n} \int_{t_n^1(1)}^{t_n^1([p/2])} |g(\tau) - g(\theta)| \frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{|\cos\tau - \cos\theta|} d\tau \right\}$$

$$+ \left| \int_0^{t_n^1(1)} (g(\tau) - g(\theta))K_1 \, d\tau \right|$$

$$\equiv J_{1,1}^1 + J_{1,1}^2 + J_{1,1}^3. \tag{40}$$

Obviously (24)–(26), $\tau \in [0, \theta/2]$, and $t_n^1([p/2]) = O(1)\theta/2$ imply $|\cos\theta - \cos\tau| = O(1)\theta^2$ and $|g(\tau) - g(0)| \leqslant \omega(g, \theta)$.

Consequently, by (12)

$$J_{1,1}^1 \leqslant O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2) \int_{t_n^1(1)}^{t_n^1([p/2])} |g(\tau) - g(\theta)|$$

$$\frac{\sin^{\alpha-r+3/2}(\tau/2)\cos^{\beta-m+1/2}(\tau/2)}{\cos\tau - \cos\theta} d\tau = \theta^{r-\alpha-1/2}\theta\omega(g,\theta)\theta^{-2}\theta^{\alpha-r+3/2}$$

$$= O(1)\omega(g, \theta). \tag{41}$$

On the other hand, for every fixed $\theta \in (0, \pi/3]$, the integral part of $J_{1,1}^1$ converges to zero by Lebesgue Convergence Theorem. Moreover, the convergence is uniform for $\theta \in [\delta, \pi/3]$, where $0 < \delta < \pi/3$ is fixed.

Now let us show that $J_{1,1}^1 = o(1)$ uniformly with respect to $\theta \in [0, \pi/3]$. Indeed, let $\varepsilon > 0$ be an arbitrary fixed number. Then there exists $\delta > 0$, such that $\omega(g, \theta) \leqslant \varepsilon$ as soon as $\theta \in [0, \delta]$. Now let us take $n(\varepsilon) \in N$ so large that $J_{1,1}^1 < \varepsilon$ as soon as $n > n(\varepsilon)$ for any $\theta \in [\delta, \pi/3]$. The rest follows from (41).

Due to (12)

$$J_{1,1}^2 = O(1)\sin^{r-\alpha-1/2}(\theta/2)\cos^{m-\beta-1/2}(\theta/2)$$

$$\frac{1}{n}\int_{t_n^1(1)}^{t_n^1([p/2])} |g(\tau)| \frac{\sin^{\alpha-r+1/2}(\tau/2)\cos^{\beta-m-1/2}(\tau/2)}{\cos\tau - \cos\theta}\, d\tau$$

$$= O(1)\theta^{r-\alpha-1/2}\frac{1}{n}\theta\omega(g,\theta)\theta^{-2}\theta^{\alpha-r+1/2} = O(1)\frac{1}{n\theta}\omega(g,\theta) = o(1) \quad (42)$$

uniformly with respect to $\theta \in [t_n^1(1), \pi/3]$ by the similar argumentation presented above. $J_{1,1}^3$ is estimated analogously.

Now let us estimate expressions for J_2 and J_4. Taking into account (18) we have

$$J_2 + J_4 \leqslant \sum_{i=1}^{2}\left\{\left|\int_{t_n^1([p/2])}^{t_n^1(p-2)} (g(\tau)-g(\theta))K_i\, d\tau\right| + \left|\int_{t_n^1(p+1)}^{\pi/2} (g(\tau)-g(\theta))K_i\, d\tau\right|\right\}$$

$$\equiv J_{2,1} + J_{2,2} + J_{4,1} + J_{4,2}. \quad (43)$$

By Abel's transformation we obtain

$$\begin{aligned} J_{2,1} &= \int_{t_n^1([p/2])}^{t_n^1(p-2)} (g(\tau)-g(\theta))K_1\, d\tau \\ &= \sum_{k=[p/2]+1}^{p-2} \int_{t_n^1(k-1)}^{t_n^1(k)} [g(\tau) - g(t_n^1(k))]K_1\, d\tau \\ &\quad + \sum_{k=[p/2]+1}^{p-2} (g(t_n^1(k)) - g(\theta)) \int_{t_n^1(k-1)}^{t_n^1(k)} K_1\, d\tau \\ &= \sum_{k=[p/2]+1}^{p-2} \int_{t_n^1(k-1)}^{t_n^1(k)} [g(\tau) - g(t_n^1(k))]K_1\, d\tau \\ &\quad + \sum_{k=[p/2]+1}^{p-3} [g(t_n^1(k)) - g(t_n^1(k+1))] \int_{t_n^1([p/2])}^{t_n^1(k)} K_1\, d\tau \\ &\quad + (g(t_n^1(p-2)) - g(\theta)) \int_{t_n^1([p/2])}^{t_n^1(p-2)} K_1\, d\tau. \end{aligned} \quad (44)$$

Taking into account Lemma 1 we have

$$J_{2,1} \leqslant \sum_{k=[p/2]}^{p-2} \max_{\tau\in[t_n^i(k-1),t_n^i(k)]} |g(\tau)-g(t_n^1(k))| \int_{t_n^1(k-1)}^{t_n^1(k)} |K_1|\,d\tau$$

$$+ \sum_{k=[p/2]}^{p-3} |g(t_n^1(k))-g(t_n^1(k+1))| \left| \int_{t_n^1([p/2])}^{t_n^1(k)} K_1\,d\tau \right|$$

$$+ |g(t_n^1(p-2))-g(\theta)| \left| \int_{t_n^1([p/2])}^{t_n^1(p-2)} K_1\,d\tau \right|$$

$$\leqslant \sum_{k=[p/2]}^{p-2} |g(I_{k,n}^1)| d_{k,n}^1 + \sum_{k=[p/2]}^{p-2} |g(t_n^1(k))-g(t_n^1(k+1))| \nabla_{k,n}^1$$

$$+ O(1)\omega\left(g,\frac{1}{n}\right)\nabla_{p-2,n}^1$$

$$\leqslant O(1)\left\{\omega\left(g,\frac{1}{n}\right) + \sum_{k=1}^{p-2} |g(I_{k,n}^1)| \frac{1}{p-k}\right\} \leqslant O(1)\left\{o(1) + \sum_{k=1}^{n} \frac{|g(I_{n_k,n}^1)|}{k}\right\}. \quad (45)$$

Analogously, we have

$$J_{4,1} = \int_{t_n^1(p+1)}^{\pi/2} (g(\tau)-g(\theta))K_1\,d\tau$$

$$= \sum_{k=p+2}^{\bar{n}} \int_{t_n^1(k-1)}^{t_n^1(k)} [g(\tau)-g(t_n^1(k))]K_1\,d\tau + \sum_{k=p+2}^{\bar{n}-1} [g(t_n^1(k+1))-g(t_n^1(k))] \int_{t_n^1(k)}^{\pi/2} K_1 d\tau$$

$$+ (g(t_n^1(p+2))-g(\theta)) \int_{t_n^1(p+1)}^{\pi/2} K_1\,d\tau$$

$$\leqslant \sum_{k=p+2}^{\bar{n}} \max_{\tau\in[t_n^1(k-1),t_n^1(k)]} |g(\tau)-g(t_n^1(k))| \int_{t_n^1(k-1)}^{t_n^1(k)} |K_1|\,d\tau$$

$$+ \sum_{k=p+2}^{\bar{n}-1} |g(t_n^1(k+1))-g(t_n^1(k))| \left| \int_{t_n^1(k)}^{\pi/2} K_1\,d\tau \right|$$

$$+\ |g(t_n^1(p+2)) - g(\theta)|\left|\int_{t_n^1(p+1)}^{\pi/2} K_1\, d\tau\right|$$

$$\leqslant \sum_{k=p+2}^{\bar{n}} |g(I_{k,n}^1)| d_{k,n}^1 + \sum_{k=p+2}^{\bar{n}-1} |g(I_{k+1,n}^1)|\, \triangle_{k,n}^1 + O(1)\omega\left(g,\frac{1}{n}\right)\triangle_{p+2,n}^1$$

$$= O(1)\left\{\omega\left(g,\frac{1}{n}\right) + \sum_{k=p+2}^{\bar{n}} \frac{|g(I_{k,n}^1)|}{k-p}\right\} \leqslant O(1)\left\{o(1) + \sum_{k=1}^{n} \frac{|g(I_{n_k,n}^1)|}{k}\right\}. \tag{46}$$

$J_{2,2}$ and $J_{4,2}$ are estimated analogously, so we omit the details. Combination of (36)–(46) completes the proof. ∎

LEMMA 3 (Sablin [14, Theorem 1, p. 88]). *Let a sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$, *satisfying the conditions of Definition* 2, *be such that* $\lim_{k\to\infty} \lambda_k/\lambda_{2k}$ *exists. Then* $\Lambda BV = \Lambda_C BV$ *if and only if this limit is less than one.*

4. PROOFS

Proof of Theorem 1. *Sufficiency*: We assume that $x \in [0,1]$ since the case $x \in [-1,0]$ is reduced to the previous one via identity (11).

Let $x \in [0,1/2]$. Then

$$\begin{aligned}|S_n^{(\alpha,\beta)}(f,x) - f(x)| &\leqslant |S_n^{(\alpha,\beta)}(f,x) - S_n^{(-1/2,-1/2)}(f,x)|\\ &\quad + |S_n^{(-1/2,-1/2)}(f,x) - f(x)|\\ &= |S_n^{(\alpha,\beta)}(f,x) - S_n(g,\theta)|\\ &\quad + |S_n(g,\theta) - g(\theta)| \equiv J_1 + J_2.\end{aligned} \tag{47}$$

$J_1 = o(1)$ as $n \to \infty$ uniformly with respect to $x \in [0,1/2]$ [5, Note, p. 179]. And conditions (4), (6), and (7) (see Remark) instantly imply the uniform convergence of $J_2 = o(1)$ on the segment $[0,1/2]$.

Now, let $x \in [1/2,1]$. It is known [8, Proof of Theorem, p. 904] that the following representation

$$S_n^{(\alpha,\beta)}(f,x) = S_{1,n}(f,x) + o(1) \tag{48}$$

is valid for $f \in C \cap EP^{(\alpha,\beta)}$, $x \in [-1,1]$, and $n \geqslant r+m$, where $S_{1,n}(f,x) \equiv (1-x)^r(1+x)^m S_{n-r-m}^{(\alpha+r,\beta+m)}(f(x)(1-x)^{-r}(1+x)^{-m},x)$.

It is obvious (see (34)) that

$$|g(I^i_{k,n})| = O(1)\omega\left(g, \frac{1}{n}\right) \leqslant O(1)\omega\left(f, \frac{1}{n}\right) \tag{49}$$

for $i = 1, 2$.

Next, due to (1) and (49), applying Abel's transformation to (35), we obtain

$$\begin{aligned} |S_{1,n}(f - f(x), x)| &= O(1)\left\{o(1) + \sum_{i=1}^{2}\sum_{k=1}^{\ell}\frac{|g(I^i_{n_k,n})|}{k} + \sum_{i=1}^{2}\sum_{k=\ell+1}^{n}\frac{|g(I^i_{n_k,n})|}{k}\right\} \\ &= O(1)\left\{o(1) + \omega\left(f, \frac{1}{n}\right)\sum_{k=1}^{\ell}\frac{1}{k} + \sum_{i=1}^{2}\sum_{k=\ell+1}^{n-1}\frac{1}{k^2}\sum_{j=1}^{k}|g(I^i_{n_j,n})|\right. \\ &\quad \left. + \sum_{i=1}^{2}\frac{1}{n}\sum_{k=1}^{n}|g(I^i_{n_k,n})|\right\}. \end{aligned} \tag{50}$$

Since $I^i_{k,n} \cap I^i_{s,n} = \emptyset$ for $k \neq s$, $i = 1, 2$, and $n \in N$, it follows from Definition 4 that

$$\sum_{j=1}^{k}|g(I^i_{n_j,n})| \leqslant \upsilon(g, k) = \upsilon(f, k)$$

and hence

$$|S_{1,n}(f - f(x), x)| = O(1)\left\{o(1) + \omega\left(f, \frac{1}{n}\right)\sum_{k=1}^{\ell}\frac{1}{k} + \sum_{k=\ell+1}^{n-1}\frac{\upsilon(f, k)}{k^2} + \frac{\upsilon(f, n)}{n}\right\}.$$

However, $\upsilon(f, n)/n = O(1)\omega(f, 1/n)$ [9, Theorem 4, p. 68]. Consequently, in view of (4) and (48), $x \in [1/2, 1]$, and the arbitrariness of $\ell \in N$ the proof is completed. ∎

Proof of Theorem 2. *Sufficiency*: Again, due to (47) and (48) it suffices to estimate the expression $|S_{1,n}(f - f(x), x)|$ for $x \in [1/2, 1]$. Without loss of generality, we may assume that $f \neq const$. Now let us set $\ell = [1/\omega(f, 1/n)]$. Then according to (50) we get

$$\begin{aligned} |S_{1,n}(f - f(x), x)| &= O(1)\left\{o(1) + \omega\left(f, \frac{1}{n}\right)\sum_{k=1}^{\ell}\frac{1}{k} + \sum_{i=1}^{2}\sum_{k=\ell+1}^{n}\frac{|g(I^i_{n_k,n})|}{k}\right\} \\ &\equiv o(1) + J_1 + J_2. \end{aligned}$$

It is clear that $J_1 = O(1)\omega(f, 1/n)\ln[1/\omega(f, 1/n)] = o(1)$ as $n \to \infty$. Taking into account that the intervals $I^i_{k,n}$ are non-overlapping and $|g(I^i_{n_k})| \geqslant |g(I^i_{n_{k+1}})|$, $k = 1, 2, \ldots$, we can estimate J_2 as follows (see Definition 3):

$$J_2 = \sum_{i=1}^{2} \sum_{k=\ell+1}^{n} \frac{|g(I^i_{n_k,n})|}{k} = \sum_{i=1}^{2} \sum_{k=1}^{n-\ell} \frac{|g(I^i_{n_{k+\ell},n})|}{k+\ell}$$

$$\leqslant \sum_{i=1}^{2} \sum_{k=1}^{n-\ell} \frac{|g(I^i_{n_k,n})|}{k+\ell} \leqslant 2\upsilon_{H(\ell)}(g) = 2\upsilon_{H(\ell)}(f) = o(1),$$

as $\ell \to \infty$, since the sequence $\lambda = (k)_{k=1}^{\infty}$ satisfies the conditions of Lemma 3. ∎

Proof of Theorem B. Since $f \in DL$,

$$\min_{1 \leqslant \ell \leqslant n} \left\{ \omega\left(f, \frac{1}{n}\right) \sum_{k=1}^{\ell} \frac{1}{k} + \sum_{k=\ell+1}^{n-1} \frac{\upsilon(k)}{k^2} \right\} \leqslant \omega\left(f, \frac{1}{n}\right) \sum_{k=1}^{n} \frac{1}{k}$$

$$= O(1)\omega\left(f, \frac{1}{n}\right) \ln n = o(1).$$

The rest follows from Theorem 1. ∎

Proof of Corollary 1. Let us set $\ell = [\exp(\ln n/(\ln\ln n)^{\gamma+1})]$. Then

$$\min_{1 \leqslant \ell \leqslant n} \left\{ \omega\left(f, \frac{1}{n}\right) \sum_{k=1}^{\ell} \frac{1}{k} + \sum_{k=\ell+1}^{n-1} \frac{\upsilon(k)}{k^2} \right\}$$

$$\leqslant O(1)\left(\omega\left(\frac{1}{n}\right) \frac{\ln n}{(\ln\ln n)^{\gamma+1}} + \sum_{k=\ell+1}^{n-1} \frac{1}{k \ln k \ln\ln k} \right)$$

$$\leqslant O(1)\left(\frac{1}{\ln\ln n} + \ln\ln\ln n - \ln \frac{\ln\ln n}{(\ln\ln n)^{\gamma+1}} \right) = o(1).$$

It is possible (see [10, p. 483]) to construct an example of a function which satisfies the condition of Corollary 1, but does not satisfy those of Theorem B or Corollary 2. ∎

Proof of Corollary 2. *Sufficiency*: If a modulus of variation $\upsilon(n)$ satisfies (6), then $V[\upsilon] \subset HBV$ [3, Theorem 2, p. 232]. Hence, sufficiency of condition (6) immediately follows from Theorem 2.

Now let us show the necessity of conditions (4), (6), and (7). Due to (47)

$$\|S_n^{(\alpha,\beta)}(f,\cdot) - S_n(g,\cdot)\|_{[a,b]} = o(1) \tag{51}$$

for $[a,b] \subset (-1,1)$ and every continuous function f.

However, it is known (see [10, Theorem 4 and Corollary 7, p. 493; 18, Theorem 3, p. 112]) that conditions (4), (6), and (7) are necessary for the convergence of Fourier series, thus by (51) they are also necessary for the convergence of Fourier–Jacobi series. ∎

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